1.3 The plotting of polynomial fractions

Plot the graph of the function $f = \dfrac{x^5 + 8x^2 - 2x - 6}{x^5 + 1}$. *The graph should display the behaviour of the f clearly.*

The task seems a little bit indefinite but soon we will understand it. It is sure that the f has to be plotted by the **plot** command. The second parameter of plot is the interval to be plotted. The question is that in which interval the function f should be plotted? The roots of the f should appear in the graph so we have to choose an interval which contains all the roots of the f. Let's determine the roots.

> $f := \dfrac{x^5 + 8x^2 - 2x - 6}{x^5 + 1}$

$$f := \frac{x^5 + 8x^2 - 2x - 6}{x^5 + 1} \tag{1}$$

> $solve(f, x)$

$$\mathrm{RootOf}(\_Z^5 + 8\,\_Z^2 - 2\,\_Z - 6, index = 1), \mathrm{RootOf}(\_Z^5 + 8\,\_Z^2 - 2\,\_Z - 6, index = 2), \mathrm{RootOf}(\_Z^5 + 8\,\_Z^2 - 2\,\_Z - 6, index = 3), \mathrm{RootOf}(\_Z^5 + 8\,\_Z^2 - 2\,\_Z - 6, index = 4), \mathrm{RootOf}(\_Z^5 + 8\,\_Z^2 - 2\,\_Z - 6, index = 5) \tag{2}$$

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The response is that the roots of the f are the roots of its own numerator. We would have known this without the help of Maple, I guess. But there is more to this than meets the eye. The **solve** command has not found a closed formula for the roots of the polynomial located in the numerator. This has been acceptable since Galois because according to the results of his theory, there are no existing closed formulas for fifth or even higher degree polynomials. Maple uses the **RootOf** notation for the representation of the algebraic numbers so that everything could go on smoothly.

So we have not received a closed formula for the roots of the f. But would we really need it? No, we would not because we want to determine only one interval that contains all the roots. To do this, it is enough to know the real approximation of the roots. The **fsolve** procedure does this.

> $fsolve(f, x)$

$$-1.953178620 \tag{3}$$

We have received the real approximation of one root as a response. Unfortunately this does not rule out the existence of more roots. While **fsolve** gives all the real approximations of all the roots for the polynomials, in the case of the rational functions the command only calculates the approximate value of one solution. But **fsolve** can do better if we help it. If we give the interval which contains a root then fsolve can determine the real approximation of the root of that interval. Let's try if it gives a root and if yes what root it gives in the interval $[-1, 0]$]. To do this, give the $x = -1\,..0$ option as a third parameter to the **fsolve** procedure.

> $fsolve(f, x, x = -1\,..0)$

$$-.7690029727 \tag{4}$$

We have received a root in the appointed interval. Naturally, we cannot try out all the possible intervals but Maple can help us with the **realroot** procedure. It isolates the roots of the univariate polynomial that was given as a parameter. First, however, separate the numerator of the fraction function f with the $numer(f)$ command.

> $n := numer(f)$

$$n := x^5 + 8\,x^2 - 2\,x - 6 \qquad \textbf{(5)}$$

> $realroot(\%)$

$$[[0, 4], [-1, 0], [-2, -1]] \qquad \textbf{(6)}$$

The output of **realroot** is our first encounter with the **list** data type of Maple. Syntactically this data type is the sequence of Maple objects between square brackets.

> $lista := \left[12, \sin(5\,x), \dfrac{x^2}{\ln(y)}, elem\right]$

$$lista := \left[12, \sin(5\,x), \frac{x^2}{\ln(y)}, elem\right] \qquad \textbf{(7)}$$

> $lista_3, lista_4; lista$

$$\frac{x^2}{\ln(y)}, elem$$

$$\left[12, \sin(5\,x), \frac{x^2}{\ln(y)}, elem\right] \qquad \textbf{(8)}$$

The examples illustrate that we can refer to the elements of the list just like in the case of the expression sequence. Do we remember how to switch to subscript? By pressing the underscore after the name of the list the cursor switches into subscript which we can exit with the **right arrow** key.

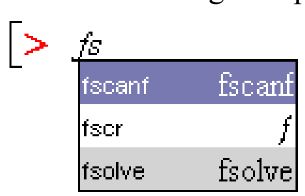
If we do not type an index after the name of the list then we get the whole list.

So, the output of **realroot** is a list containing three elements and each element is a two-element list. The two-element lists determine such open intervals in which the polynomial f has exactly one root. After this we can be sure that the three real roots of the f are:

> $fsolve(f, x, x = -2\,..-1),\ fsolve(f, x, x = -1\,..0),\ fsolve(f, x, x = 0\,..4)$

$$-1.953178620,\ -.7690029727,\ 0.9445420160 \qquad \textbf{(9)}$$

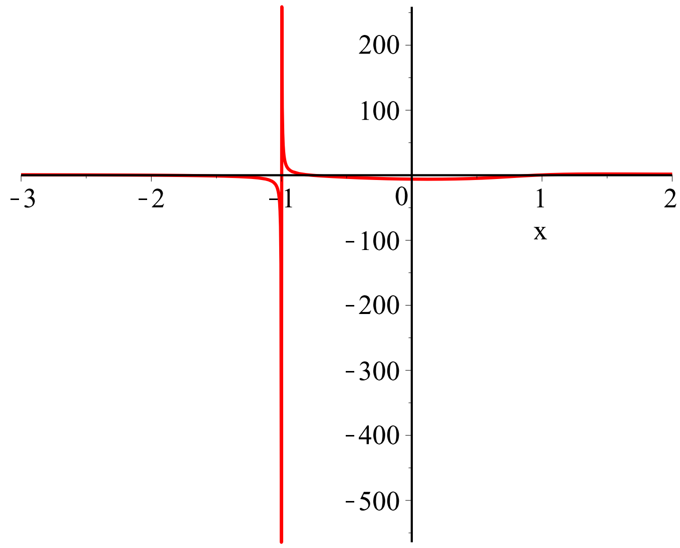
We have typed **fsolve** so many times that we should have realised the importance of the instruction completion functionality of Maple. This enables the user not to type the whole name of the procedures. It is enough if we type only the first letters of the desired procedure and press **CTRL+SPACE**. This makes a dropdown list appear which contains the names of those procedures which match the typed characters. We can choose from this list the usual way (e.g. with mouse click). For example, if we press the *fs* characters and after **CTRL+SPACE** then the following list appears on the screen.



If we click on **fsolve** the system completes the **fs** character sequence to **fsolve**. It is recommended to experiment and get used to using the instruction completion functionality. Naturally in this case the best solution is the golden mean. If we give few initial characters (e.g. only one) then we might have to choose from a really long list, which can be tiring. If we type more and more characters to make the list shorter, then it might be easier to finish the name of the procedure by typing rather than choosing from a list that contains only some elements.

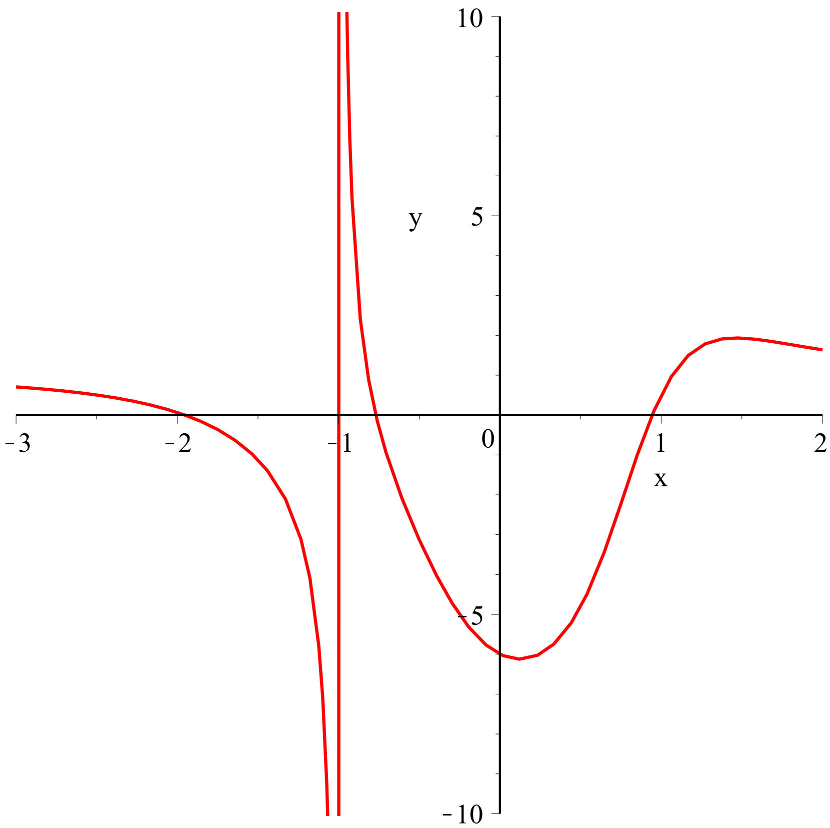
So far we have learnt that it is recommended that the left endpoint of the interval is a little bit smaller than the smallest root and the right endpoint is a little bit bigger than the largest root.

> $plot(f, x = -3..2)$



This graph does not seem so fine. The reason for this is that the value of the f is getting bigger and bigger while getting closer to the $x = -1$ point from the right. Notice that the scale of the y axis finishes above 200 while in most parts of the graph the values lower than five are dominant. In this case, the option of the **plot** command that can be given as a third argument comes in handy. This option determines the domain for the values of the function.

> $plot(f, x = -3..2, y = -10..10)$



We can be surprised when we see the details of the behaviour of the function. If we look at the graph it seems if the function had an extreme in the $[1, 2]$ interval. Examine the possible extremes of the f, that is, those x-es for which the derivative of the f is 0.

> $\frac{d}{dx} f$

$$\frac{5x^4 + 16x - 2}{x^5 + 1} - \frac{5(x^5 + 8x^2 - 2x - 6)x^4}{(x^5 + 1)^2} \tag{10}$$

> $solve(\mathbf{(10)}, x)$

$$\mathrm{RootOf}(-35\_Z^4 + 24\_Z^6 - 16\_Z - 8\_Z^5 + 2, index = 1), \mathrm{RootOf}(-35\_Z^4 + 24\_Z^6 - 16\_Z - 8\_Z^5 + 2, index = 2), \mathrm{RootOf}(-35\_Z^4 + 24\_Z^6 - 16\_Z - 8\_Z^5 + 2, index = 3), \mathrm{RootOf}(-35\_Z^4 + 24\_Z^6 - 16\_Z - 8\_Z^5 + 2, index = 4), \mathrm{RootOf}(-35\_Z^4 + 24\_Z^6 - 16\_Z - 8\_Z^5 + 2, index = 5), \mathrm{RootOf}(-35\_Z^4 + 24\_Z^6 - 16\_Z - 8\_Z^5 + 2, index = 6) \tag{11}$$

> $fsolve(\mathbf{(10)}, x)$

$$0.1244656572 \tag{12}$$

> $fsolve((\mathbf{10}), x, x = 1..2)$

$$1.463114340 \tag{13}$$

> $\frac{d^2}{dx^2} f$

$$\frac{20 x^3 + 16}{x^5 + 1} - \frac{10 (5 x^4 + 16 x - 2) x^4}{(x^5 + 1)^2} + \frac{50 (x^5 + 8 x^2 - 2 x - 6) x^8}{(x^5 + 1)^3} - \frac{20 (x^5 + 8 x^2 - 2 x - 6) x^3}{(x^5 + 1)^2} \tag{14}$$

> $signum(eval((\mathbf{14}), x = (\mathbf{13})))$

$$-1 \tag{15}$$

A brief explanation is needed here. We got the derivative of the f but then we did not succeed in specifying the roots with a closed formula. That is why we have used the **fsolve** procedure which has given the approximation of the root between 0 and 1. But we need the root in the $[1, 2]$ interval which we have got by giving the $x = 1..2$ option in the call of **fsolve**.

Notice that the derivative was not created with the `diff(f,x)`command but with the usage of the $\frac{d}{dx} f$ notation commonly used in calculus. This happened because we have chosen the $\frac{d}{dx} f$ placeholder from the **Expression** palette. The same was done when calculating the second derivative. After inserting the $\frac{d}{dx} f$ placeholder into the input line, the d and x were squared. Notice that we could have got the second derivative with the `diff(f,$2)` call of **diff** as well.

The last command contains two procedures which have not been used yet. The **eval** procedure evaluates expressions while we can give what value we want to give to the variables of the expression. The call of the $eval((\mathbf{14}), x = (\mathbf{13}))$ procedure evaluates the expression **(14)**, that is, the second derivative in the $x = (\mathbf{13})$ point which is at the zero of the derivative of the function. Since we want to get only the sign of the value we can use the **signum** procedure. According to the result, the substitution value of the second derivative is negative so our function has a maximum in point **(13)**. In this case we can see that our graph and approach have been right.

The function f is continuous in the $1 < x$ interval. It has a maximum in 1.463114340 so near the right side of this point it has to decrease. Since it has got no more roots it cannot decrease in such a way that it crosses the x axis. So, how does the function behave for the values $(\mathbf{13}) < x$? The behaviour of the function in $x < -2$ is interesting as well. To see this, consider the limit value of the f in the infinity. The command below calls the procedure **limit**. As a first parameter, it gives the expression the limit value of which we are looking for. As the second parameter it gives the domain where we are looking for the limit value. **Infinity** is the notation of the infinity in Maple.

> $limit(f, x = infinity)$

$$1 \tag{16}$$

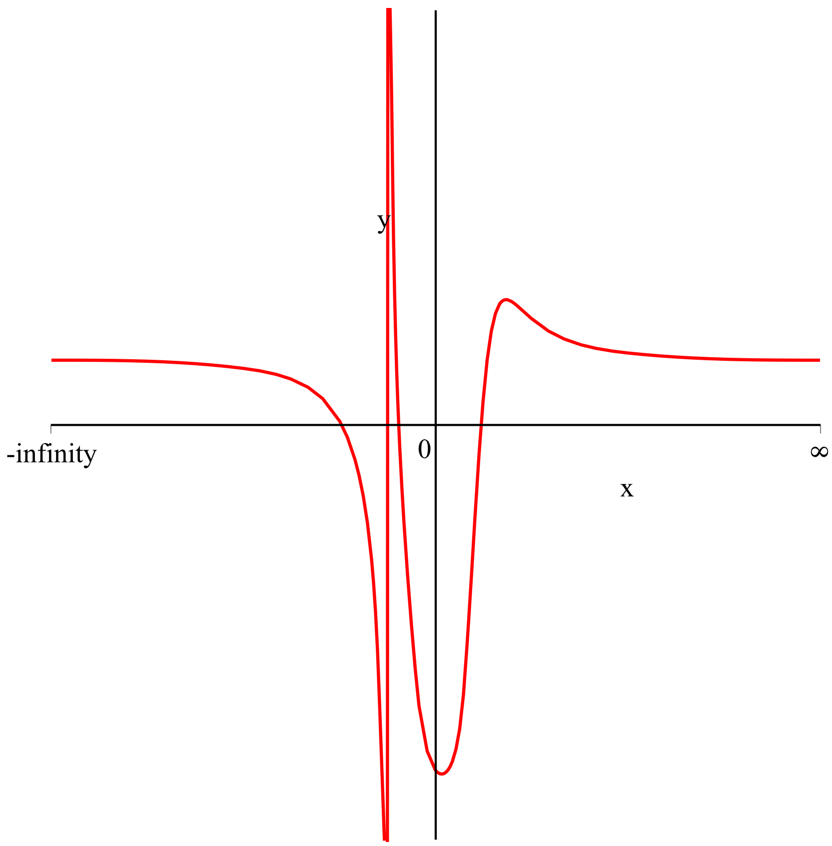
The previously experienced problem can arouse at this point: this notation is not similar to the common notation of the limit value. The solution is to use the **Expression** palette. We can find the $\lim_{x \to a} f$ expression here. We can replace the sign a with ∞ sign from the **Common Symbol** palette.

> $\lim_{x \to -\infty} f$

$$1 \tag{17}$$

According to this the function f approaches to the line $y=1$ while the x is tending to either plus or minus infinity. This cannot be seen in the previous graph. It is recommended that the limits of the plotting interval are to be chosen for the plus and minus infinity.

> $plot(f, x = -infinity..infinity, y = -10..10)$



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Let's summarise what we have done so far. First, we defined the interval which contains the roots of the f. This interval is a good solution for the domain in the case of polynomials while we should act more carefully in the case of rational functions. That is why we had to find the extremes and the limits of the function. Since both limits in plus and minus infinity existed, we had to define the plotting interval as $x = -infinity..infinity$.

What have we learnt about Maple?

• Palettes contain not only simple symbols but expression patterns as well, e.g. $\sqrt[n]{a}$. In this pattern n and a are the so-called placeholders. We can switch back and forth between the placeholders with

CTRL+TAB and **TAB**. When we input the exact values of the placeholders we can use the constants or the patterns of the palettes.

- Use **CTRL+SPACE** to activate the instruction completion functionality of Maple.
- The command **fsolve** gives the real approximation of the roots of the functions. In the case of polynomials **fsolve** approximates all the roots. In any other cases it gives only one root. As an option we can give **fsolve** the interval to which the root is to be approximated: $fsolve(f, x, x = a..b)$.
- The **numer** procedure gives the numerator of the fraction or the rational function given as a parameter. The **denom** procedure gives the denominator.
- The procedure **realroot** isolates the roots of the univariate polynomials. Its output is $[[a_1..b_1], ..., [a_n..b_n]]$ and each interval contains a root.
- The $diff(f, x\$ i)$ command gives the i^{th} derivative of the expression f.
- The **limit** procedure determines the limit values of the functions in the finite and the infinity. So $limit(f, x = a)$ is the limit value of the f while the x is approaching to a.
- In the case of the **plot** procedure we can give the domain of the function values as a third parameter. So the syntax of **plot** is: $plot(f, x = a..b, y = c..d)$.

Exercises

1. Isolate the roots of the following polynomials and find the roots in each interval.
   - $x^3 - 3x - 1$
   - $x^3 - 7x^2 - 2x - 1$
   - $x^4 - x - 1$
   - $x^4 + x^2 - 1$
   - $x^4 + 4x^3 - 12x + 9$

2. Try out the $fsolve(f, x, x = a..b)$ command on such an f and in such an interval $x = a..b$ where the f has not got a root. What is the response of the system?

3. Determine the following limits!
   - $\lim_{x \to 0} \left(\frac{\sin(x)}{x} \right)$
   - $\lim_{n \to infinity} \left(\frac{n}{3n + 2} \right)$
   - $\lim_{n \to infinity} \left(\frac{n^2 + 1}{2n + 1} - \frac{3n^2 + 1}{6n + 2} \right)$
   - $\lim_{n \to infinity} \left(\frac{n}{\sqrt{n} - \sqrt{n + 1}} \right)$

4. What is the output of $\lim_{x \to a} f$ if the limit does not exist?

5. Plot the following functions.

- $f = \tan(x); \quad x = -\pi..\pi$
- $f = \frac{\sin(x)}{x}; \quad x = -1..1$
- $f = \frac{x^2 - 5x + 4}{x^2 - 6x - 7}$
- $f = \ln\left(\frac{x^2 - 5x + 4}{x2.6x - 7}\right)$
- $f = x \sin\left(\frac{1}{x}\right)$

6. Examine the extrem values and their locations of the following functions. Plot the first and the second derivatives of the functions on one figure.

- $f = 2x^4 - 4x^3 - 11x^2 + 8x + 4$
- $f = x^5 - 5x^4 + 5x^3 + 7$
- $f = \frac{e^{(-x^2)}}{x^2}$
- $f = \sin(x) + x\cos(x)^2$
- $f = x^2 e^{(-2x)}$